Magic in Matrix Product States and Fermionic Gaussian States

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FIRST WORKSHOP ON MANY-BODY QUANTUM MAGIC (MBQM2024)

Main papers

"Quantum State Designs with Clifford Enhanced Matrix Product States"
GL, Tobias Haug, Jacopo De Nardis
arxiv:2404.18751

"The magic of free fermionic states"
Permutation[**GL**, Jacopo De Nardis, Vincenzo Alba, Mario Collura]
arxiv:2411......

Others

"Nonstabilizerness via Perfect Pauli Sampling of Matrix Product States"
GL, Mario Collura
Phys. Rev. Lett. 131, 180401

"Anticoncentration of random tensor network states"
GL, Jacopo De Nardis, Xhek Turkeshi
arxiv:2409.13023

"Estimating Non-Stabilizerness Dynamics Without Simulating It"
Alessio Paviglianiti, **GL**, Mario Collura, Alessandro Silva arxiv:2405.06054

"Clifford Dressed Time-Dependent Variational Principle"
Antonio F. Mello, Alessandro Santini, **GL**, Jacopo De Nardis, Mario Collura
arxiv:2407.01692

Jacopo De Nardis

Mario Collura

Xhek Turkeshi

Tobias Haug

Vincenzo Alba

Alessio Paviglianiti

Alessandro Silva

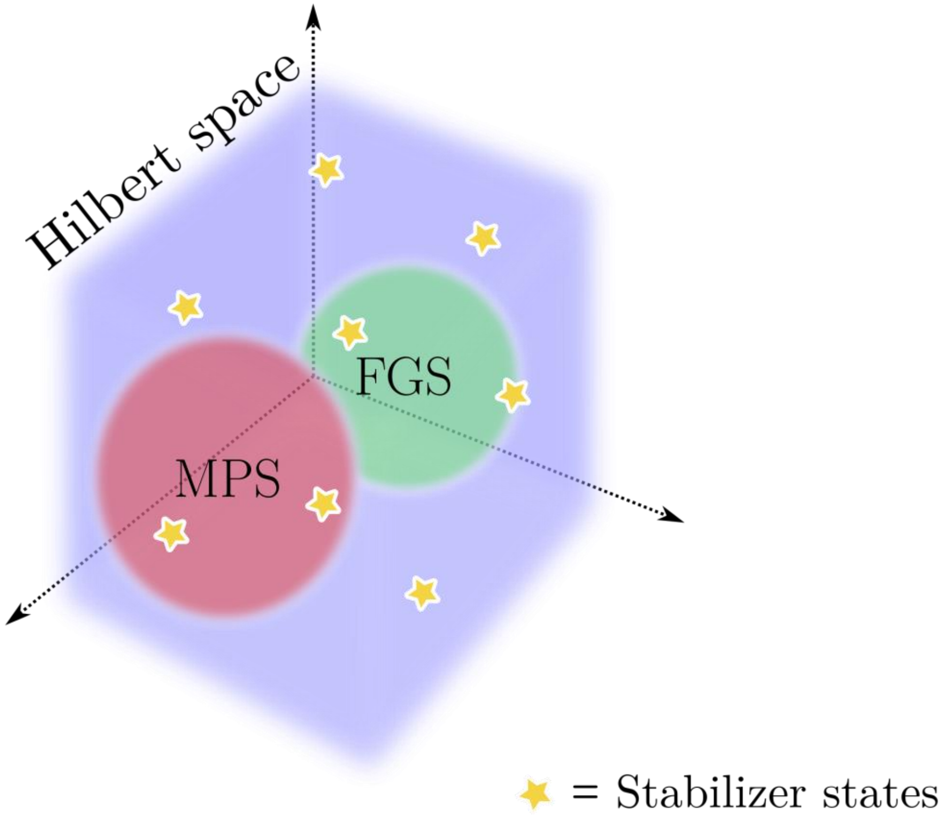
Alessandro Santini

Antonio F. Mello

Physically interesting many-body states

Which (pure) quantum states are used mostly to do many-body / condensed matter physics?

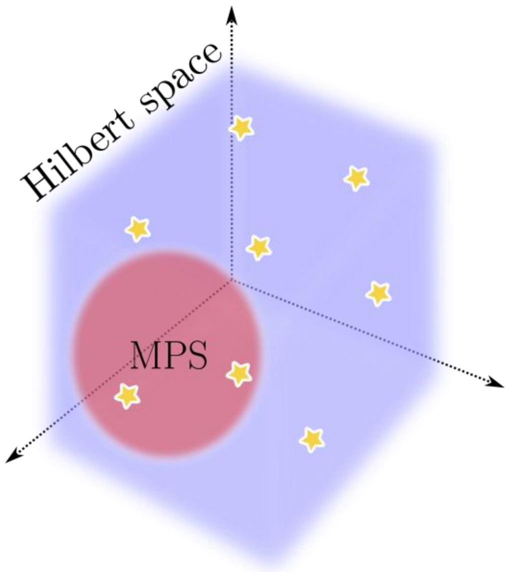
- Matrix Product States (MPS)
- Fermionic Gaussian States (FGS)

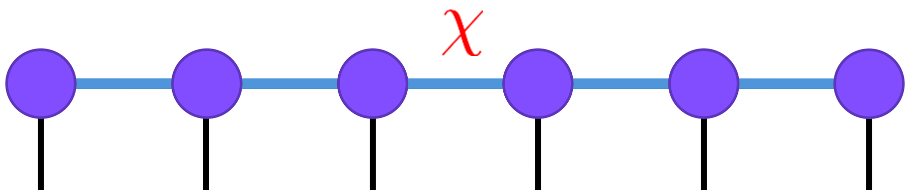


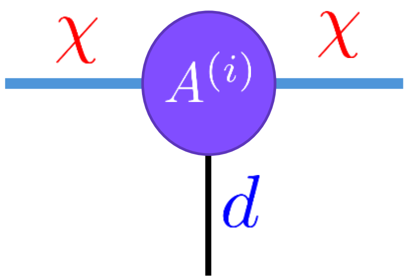
Matrix Product States (MPS)

Prototypical entangled many-body states

- ground-states in 1D are MPS with finite bond dimension χ;
- extremly useful in numerical simulations of ground states and time-evolution (DMRG, TEBD, TDVP, etc.);
- relatively easy to generate in lab (digital quantum platforms).







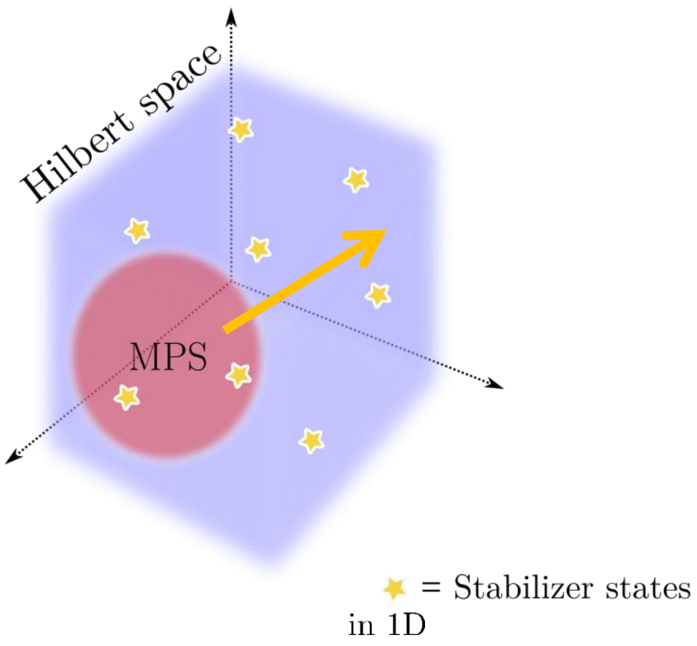
χ = bond dimension

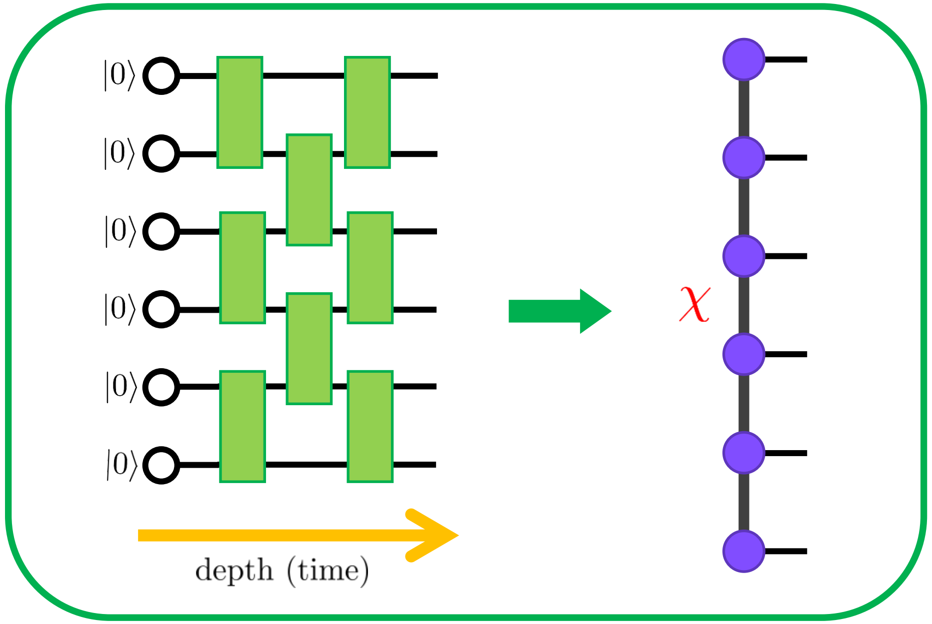
d = local physical dimension

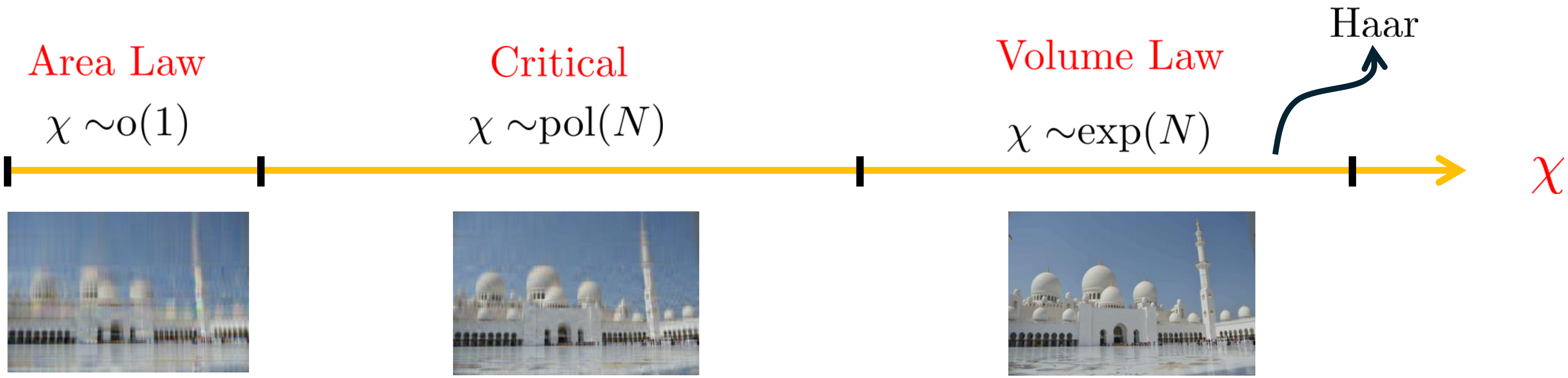
★ = Stabilizer states

U. Schollwoeck (2011)

Matrix Product States (MPS)





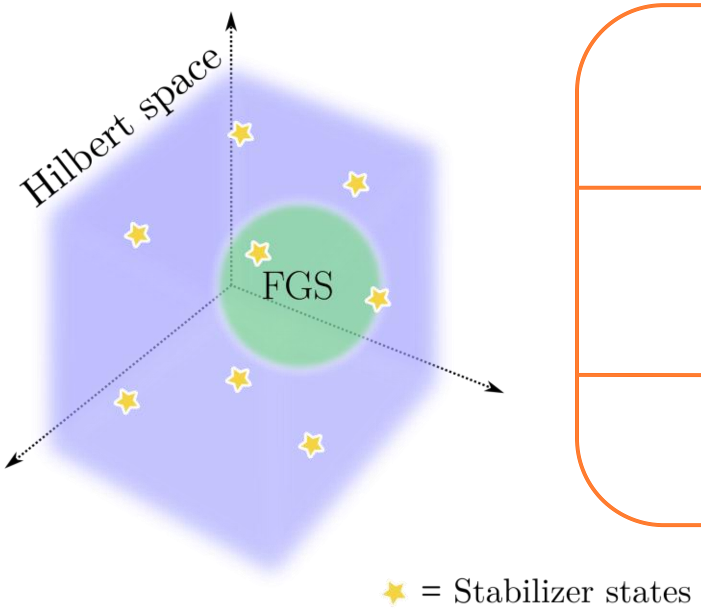


U. Schollwoeck (2011)

Fermionic Gaussian States (FGS)

Fermionic Gaussian states are a very important class of states

- fundamental in condensed matter (Slater determinants, BCS wave function, quantum chemistry)
- extremly interesting also from the quantum info point of view (matchgate circuits).



$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ $\mu, \nu = 1, 2...2L$	Canonical Anticommutation Relations
$\gamma_{2i} = \sigma_1^z...\sigma_{i-1}^z\sigma_i^x$ $\gamma_{2i+1} = \sigma_1^z...\sigma_{i-1}^z\sigma_i^y$	Jordan Wigner transformation
$\Gamma_{\mu\nu} = -\frac{i}{2}\mathrm{Tr}[[\gamma_\mu, \gamma_\nu]\rho]$	$2L \times 2L$ covariance matrix

J. Surace, L. Tagliacozzo (2022)

Main questions of this talk

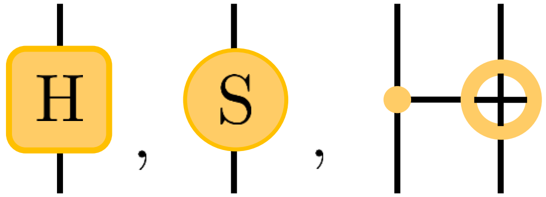
1. How much quantum recource is stored in typical MPS and FGS?
2. How can the amount of quantum resource in MPS and FGS be evaluated (numerically)?

Main questions of this talk

1. How much quantum magic is stored in typical MPS and FGS?
2. How can the amount of quantum magic in MPS and FGS be evaluated (numerically)?

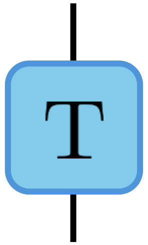
Magic (in one slide)

Clifford



CNOT

Non-Clifford



$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

$$\begin{cases} T\sigma^x T^\dagger = \frac{1}{\sqrt{2}}(\sigma^x + \sigma^y) \\ T\sigma^y T^\dagger = \frac{1}{\sqrt{2}}(-\sigma^x + \sigma^y) \\ T\sigma^z T^\dagger = \sigma^z \end{cases}$$

Universal set of gates

- Clifford gates are relatively straightforward to implement fault-tolerantly in a Quantum Error Correction code
- Non-Clifford gates can be regarded as a "quantum resource" (in a well-defined mathematical sense) → quantum magic

S. Bravyi, A. Kitaev (2004)

Stabilizer Rényi Entropies (SRE)

$$m_n(|\psi\rangle) = D^{-n} \sum_{\boldsymbol{\sigma}} \langle\psi|\boldsymbol{\sigma}|\psi\rangle^{2n} \qquad \mathcal{M}_n = (1-n)^{-1} \log m_n(|\psi\rangle) - \log D \qquad D = d^N$$

L. Leone, S. F. E. Oliviero, A. Hamma (2021)

“Quantum State Designs with Clifford Enhanced Matrix Product States”

GL, Tobias Haug, Jacopo De Nardis

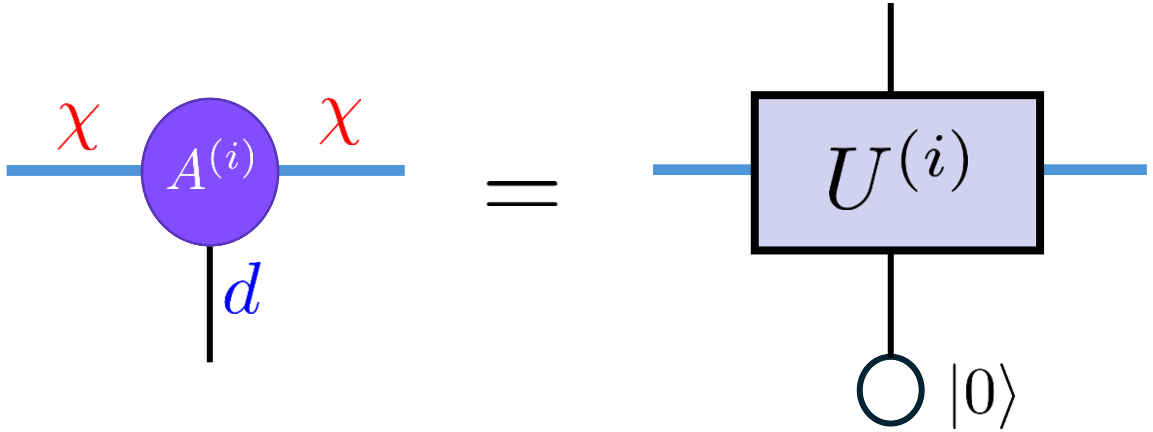
arxiv:2404.18751

1. How much quantum magic is stored in typical MPS?

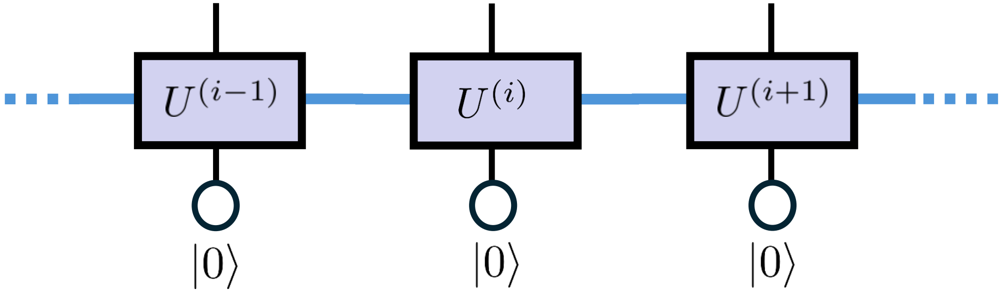
Random Matrix Product States

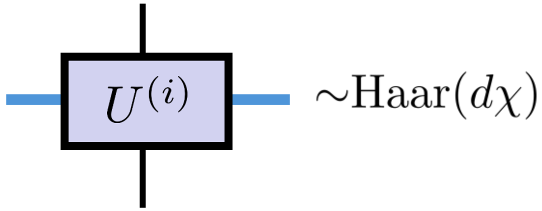
What should you expect from *typical* realizations of MPS?

MPS tensors are sub-blocks of Haar matrices:



$U^{(i)} \sim$ unitary Haar matrix of size $d\chi$





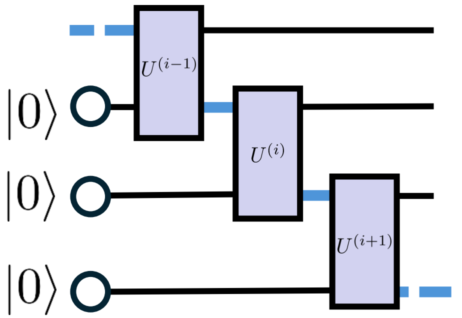
S. Garnerone, T. R. de Oliveira, P. Zanardi (2009)
J. Haferkamp, C. Bertoni, I. Roth, J. Eisert (2021)

Why Random Matrix Product States?

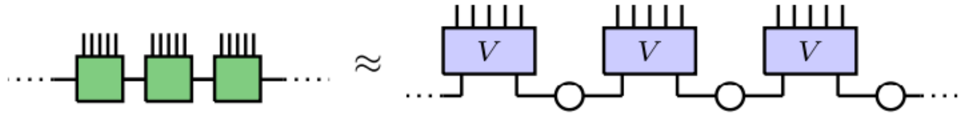
Other motivations:

- clear connection with numerics;
- RMPS as an analytically solvable random circuit;
- MPS are easy to generate in lab;
- connection with entanglement phase transition.

Quantum circuit equivalent to (R)MPS



Efficient preparation via Measurements and Feedback



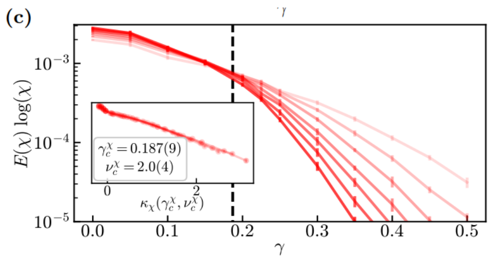
D. Malz, G. Styliaris, Z. Wei, J. I. Cirac (2024)

Y. Zhang, S. Gopalakrishnan, G. Styliaris (2024)

Measurement-induced phase transitions by matrix product states scaling

Guillaume Cecile,[1] Hugo Lóio,[1] and Jacopo De Nardis[1]

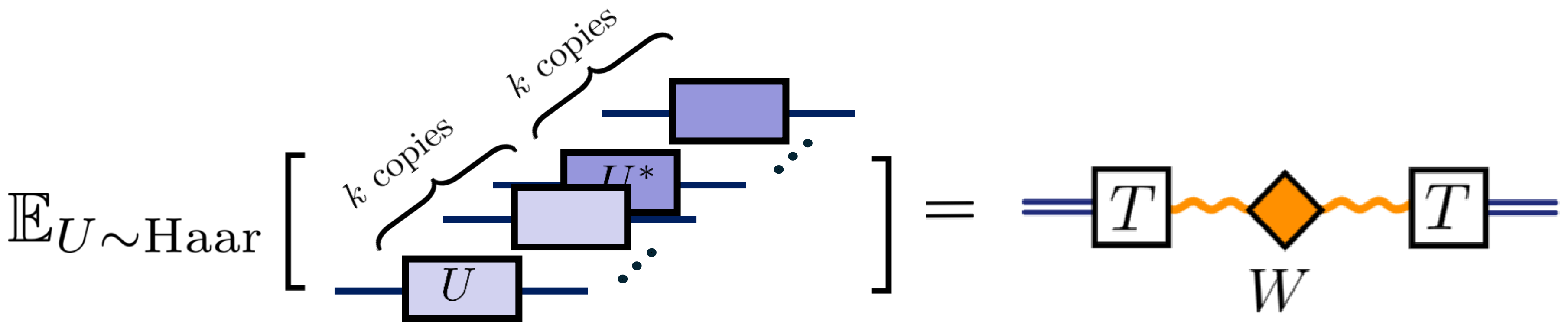
[1] *Laboratoire de Physique Théorique et Modélisation, CNRS UMR 8089, CY Cergy Paris Université, 95302 Cergy-Pontoise Cedex, France*



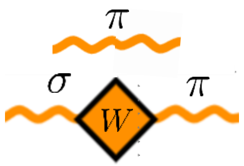
G. Cecile, H. Lóio, J. De Nardis (2023)

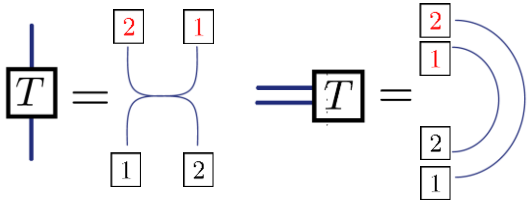
Methods: Weingarten calculus

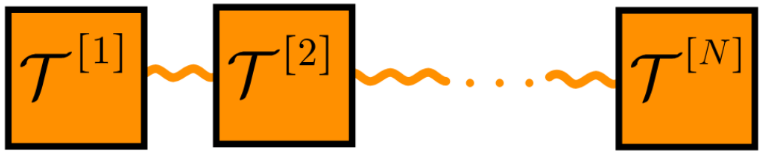
To compute average of k replicas of $U^{(i)}$,$U^{(i),*}$ we need **Weingarten calculus**!



- permutation operator (permutes k replicas)
- permutation index $\pi \in S_k$
- $\mathrm{Wg}_{\sigma\pi}$ Weingarten matrix







$\mathcal{T}$=transfer matrix in the replica (permutations) space

Magic of RMPS

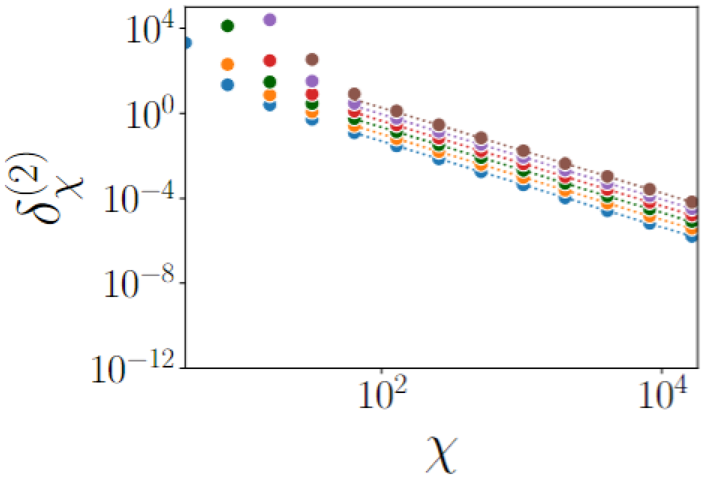
Our results

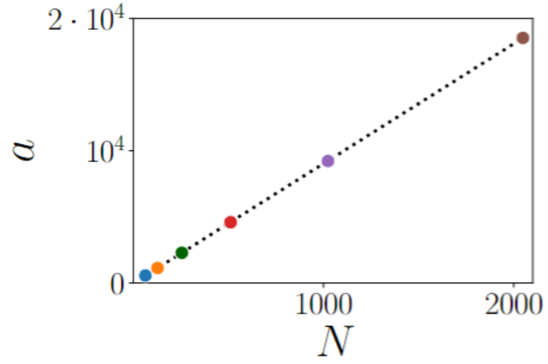
MPS with small bond dimension $\chi \sim$pol(N) can store as magic as Haar states!

Deviation of the RMPS magic from that of Haar states:

$$\delta_\chi^{(n)} = D^n\left(\mathbb{E}_{\psi\sim RMPS}[m_n(|\psi\rangle)] - \mathbb{E}_{\psi\sim Haar}[m_n(|\psi\rangle)]\right)$$

$$\delta_\chi^{(2)} = \mathcal{O}\left(\frac{N}{\chi^2}\right) \qquad \delta_\chi^{(3)} = \mathcal{O}\left(\frac{N}{\chi^3}\right)$$





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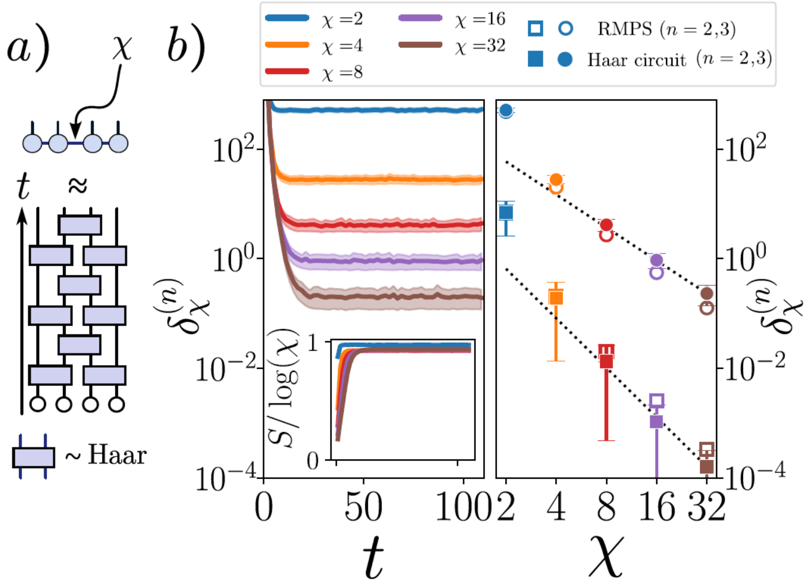
$$\delta_\chi^{(2)} = \mathcal{O}\left(\frac{N}{\chi^2}\right) \qquad \delta_\chi^{(3)} = \mathcal{O}\left(\frac{N}{\chi^3}\right)$$

Numerical benchmark (Haar circuit, MPS truncated at bond dimension χ):

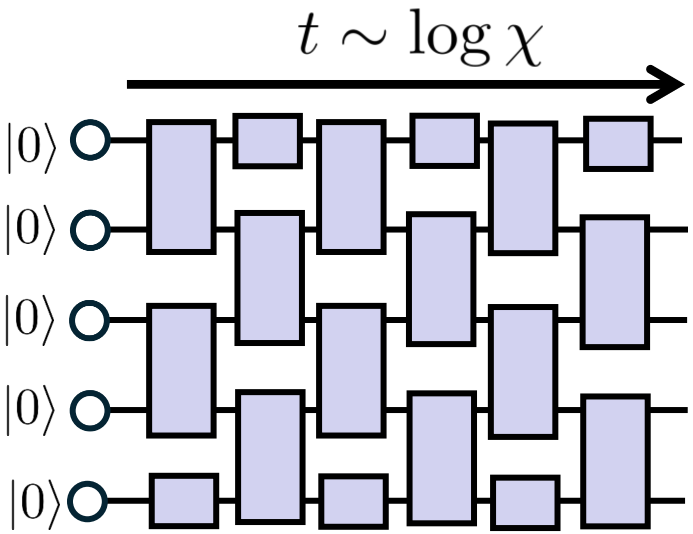


Magic of RMPS: benchmarks

Our results

MPS with small bond dimension $\chi \sim$pol(N) can store as magic as Haar states!

Consistent with other results in which SREs have been found to saturate at time $t \sim \log N$

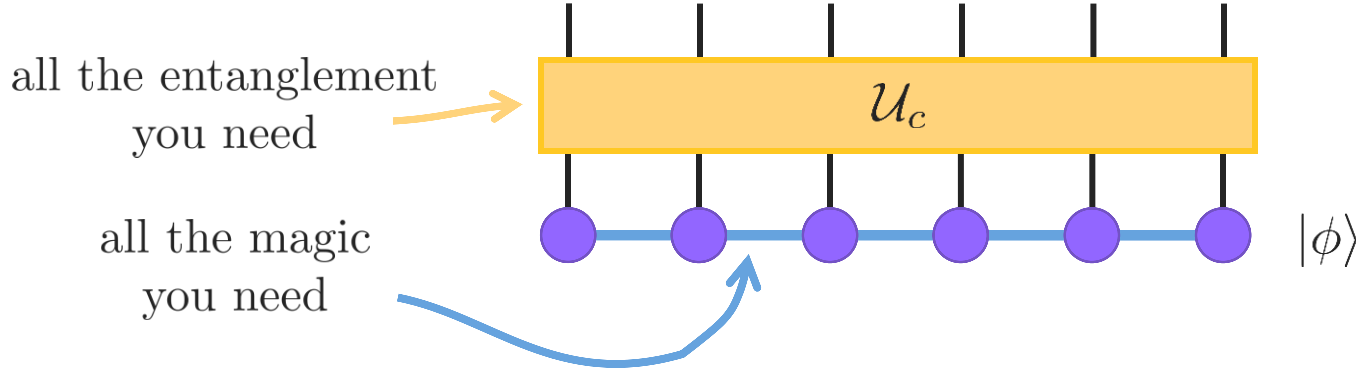


X. Turkeshi, E. Tirrito, P. Sierant (2024)

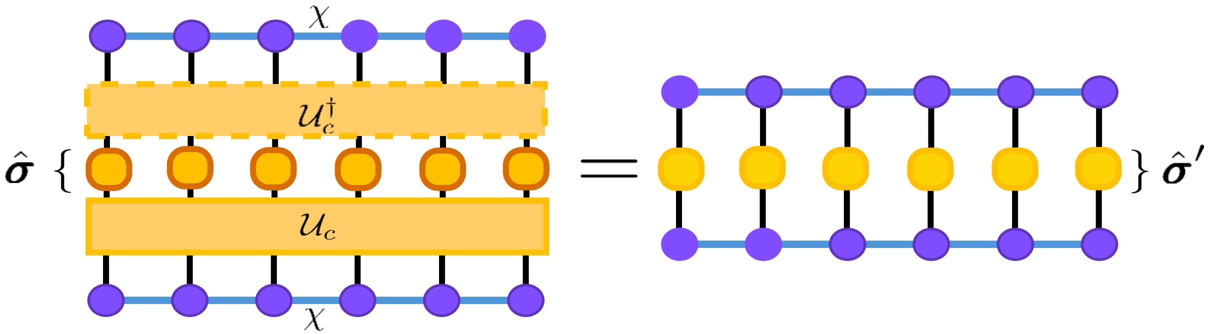
Clifford enhanced MPS ($\mathcal{C}$MPS)

We introduce the following ensemble of states:

$$\mathcal{C}\text{MPS} = \{ |\psi\rangle = \mathcal{U}_c |\phi\rangle_\chi \,, \mathcal{U}_c \in \mathcal{C}_N \text{ and } |\phi\rangle_\chi \in \text{MPS} \} \,,$$



Even if $\mathcal{C}$MPS are volume-law entangled, one can efficiently compute expectation values over them:

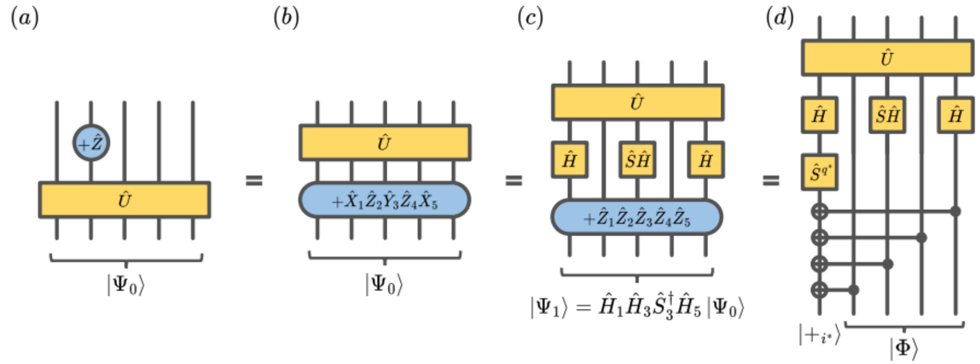


Clifford enhanced MPS: practical relevance

Recently, $\mathcal{C}$MPS algorithms for ground state, time-evolution, circuits with

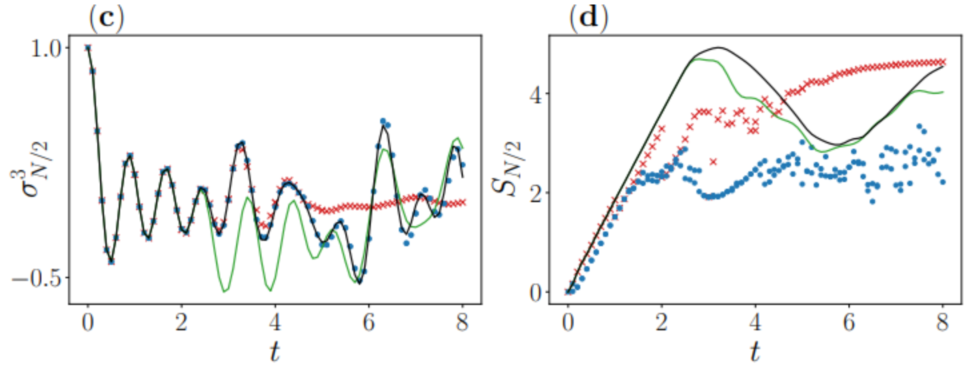
Iterative construction for Clifford circuits with measurements

A. Paviglianiti, **GL**, M. Collura, A. Silva (2024)
arXiv:2405.06054



Clifford Dressed Time-Dependent Variational Principle (TDVP)

A. F. Mello, A. Santini, **GL**,
J. De Nardis, M. Collura (2024)
arXiv:2407.01692



Similar approaches:

- "Augmented" DMRG and TDVP X. Qian, J. Huang, M. Qin (2024)
- G. E. Fux, B. Beri, R. Fazio, E. Tirrito (2024)

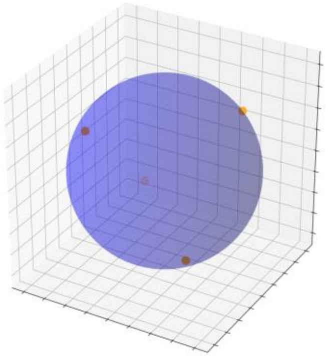
Clifford enhanced MPS: frame potential

$\mathcal{C}$MPS are arbitrarily entangled! How well do they approximate Haar states?

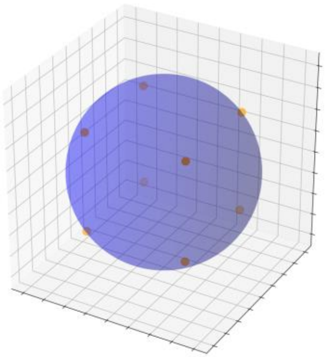
$$\mathcal{F}^{(k)} = \mathbb{E}_{\psi,\psi'}[|\langle\psi|\psi'\rangle|^{2k}] = k\text{-th Frame potential}$$

$$||\rho^{(k)} - \rho^{(k)}_{\mathrm{Haar}}||_2 = |\mathcal{F}^{(k)} - \mathcal{F}^{(k)}_{\mathrm{Haar}}| < \epsilon \Rightarrow \epsilon\text{-approximate } k\text{-design}$$

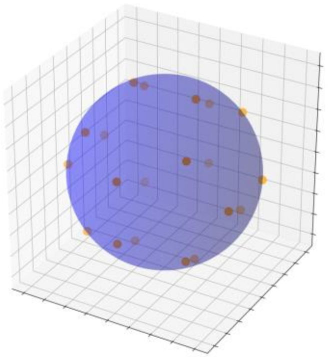
$$\rho^{(k)} = \mathbb{E}_{\psi}[|\psi\rangle\langle\psi|^{\otimes k}] \qquad \rho^{(k)}_{\mathrm{Haar}} = \mathbb{E}_{\psi\sim\mathrm{Haar}}[|\psi\rangle\langle\psi|^{\otimes k}]$$



$k = 2$



$k = 3$



$k = 5$

For $k = 1, 2, 3$: $\mathcal{F}^{(k)}_{\mathcal{C}\mathrm{MPS}} = \mathcal{F}^{(k)}_{\mathrm{Haar}}$, meananing that $\mathcal{C}$MPS are (exact) 3-design. What about $k = 4$?

$$\Delta^{(4)} = \left(\frac{\mathcal{F}^{(4)}_{\mathcal{C}\mathrm{MPS}} - \mathcal{F}^{(4)}_{\mathrm{Haar}}}{\mathcal{F}^{(4)}_{\mathrm{Haar}}}\right) \sim \frac{1}{2D}\delta^{(2)}_{\chi}$$

"Nonstabilizerness via Perfect Pauli Sampling of Matrix Product States"
GL, Mario Collura
Phys. Rev. Lett. 131, 180401

"The magic of free fermionic states"
Permutation[**GL**, Jacopo De Nardis, Vincenzo Alba, Mario Collura]
arxiv:2411......

2. How can the amount of quantum magic in MPS and FGS be evaluated (numerically)?

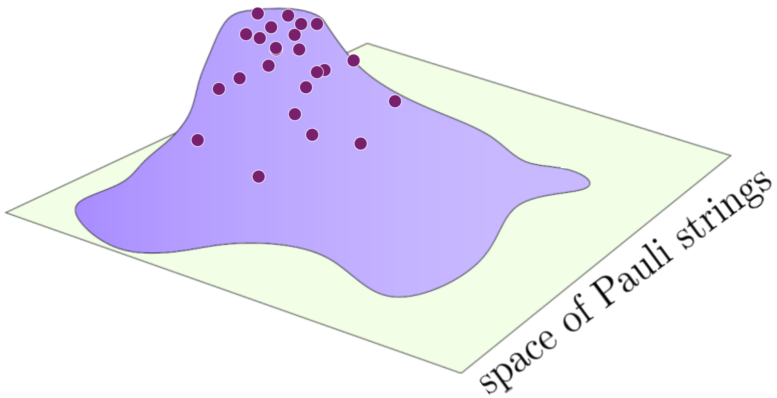
SRE by sampling

$$\Pi_\psi(\boldsymbol{\sigma}) = \frac{1}{D}\langle\psi|\boldsymbol{\sigma}|\psi\rangle^2 \qquad m_n(|\psi\rangle) = \mathbb{E}_{\boldsymbol{\sigma}\sim\Pi_\psi(\boldsymbol{\sigma})}[\Pi_\psi^{n-1}]$$

- Sample Pauli strings $\boldsymbol{\sigma}$ with probability $\Pi_\psi(\boldsymbol{\sigma})$
- Given a list of samples $\{\boldsymbol{\sigma}^k\}_{k=1}^{\mathcal{N}}$ estimate:

$$\begin{cases} \mathcal{M}_1 \simeq -\frac{1}{\mathcal{N}}\sum_{k=1}^{\mathcal{N}} \log \Pi_\psi(\boldsymbol{\sigma}^k) - \log D & n=1 \\ \mathcal{M}_n \simeq (1-n)^{-1} \log\left(\frac{1}{\mathcal{N}}\sum_{k=1}^{\mathcal{N}} \Pi_\psi^{n-1}(\boldsymbol{\sigma}^k)\right) - \log D & n>1 \end{cases}$$

statistical estimator



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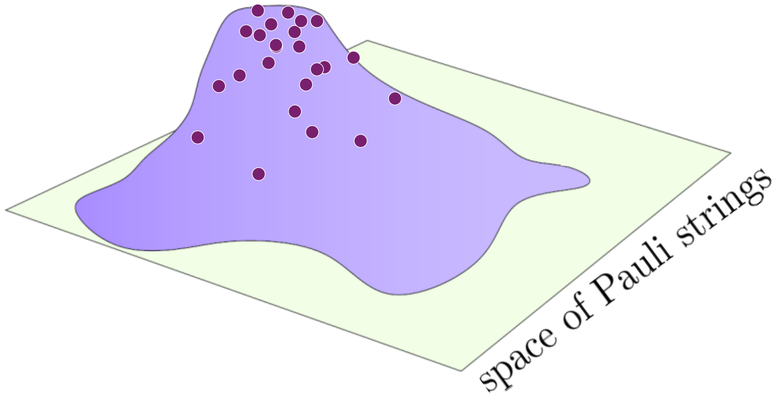
$$\Pi_\psi(\boldsymbol{\sigma}) = \frac{1}{D}\langle\psi|\boldsymbol{\sigma}|\psi\rangle^2 \qquad m_n(|\psi\rangle) = \mathbb{E}_{\boldsymbol{\sigma}\sim\Pi_\psi(\boldsymbol{\sigma})}[\Pi_\psi^{n-1}]$$

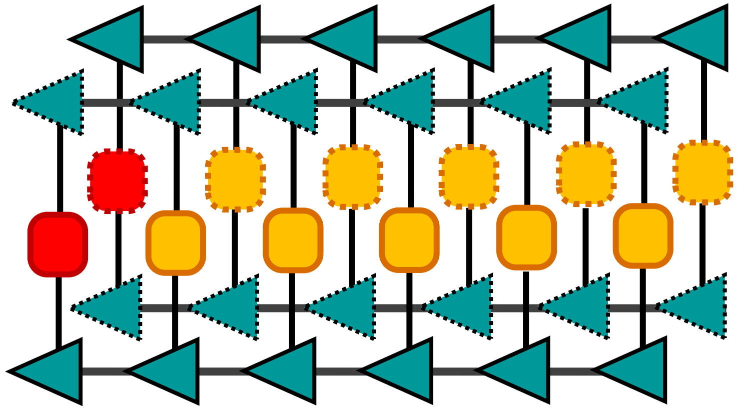
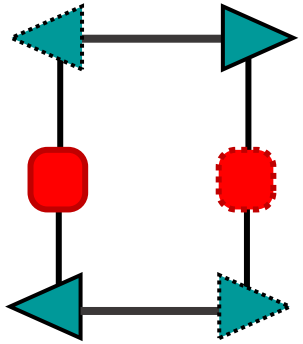
Benefits / issues

- simple, efficient, and highly parallelizable ($\mathcal{N} \sim 10^7$)
- for $n = 1$, $\mathcal{N} \sim N$ is enough to reach a given accuracy
- for $n > 1$, in worst case scenario, $\mathcal{N} \sim \exp(N)$ to reach given accuracy
- this does not necessarily occur, particularly with 'atypical' states (low-energy spectrum)



Pauli sampling of MPS

$$\Pi_\rho(\boldsymbol{\sigma}) = \pi_\rho(\sigma_1)\pi_\rho(\sigma_2|\sigma_1)\cdots\pi_\rho(\sigma_N|\sigma_1\cdots\sigma_{N-1})$$

$$\pi(\sigma_1) = \sum_{\sigma_2\ldots\sigma_N} \Pi_\rho(\boldsymbol{\sigma}) = \frac{1}{2^N}\sum_{\sigma_2\ldots\sigma_N}$$  $$=$$ 

IMPORTANT: it is a "perfect" sampler: no Markov chain Monte Carlo!

Majorana sampling of FGS

Our results

There exists an algorithm that allows to sample Pauli (Majorana) strings efficiently with probability:

$$\Pi_\psi(x_1, x_2 ... x_N) \propto |\langle\psi|(\gamma_1)^{x_1}(\gamma_2)^{x_2}...(\gamma_{2L})^{x_{2L}}|\psi\rangle|^2 = \det[\Gamma|_{x_1...x_{2L}}]$$

$$x_i \in \{0, 1\}$$

The Algorithm is efficient $O(L^3)$ per sample

IMPORTANT: it is a "perfect" sampler: no Markov chain Monte Carlo!

Technical remark: this is a Determinantal Point Process (DPP)
sampling is achieved by using particular formulas on (partial) sums of minors

"The magic of free fermionic states"

Permutation[**GL**, Jacopo De Nardis, Vincenzo Alba, Mario Collura]

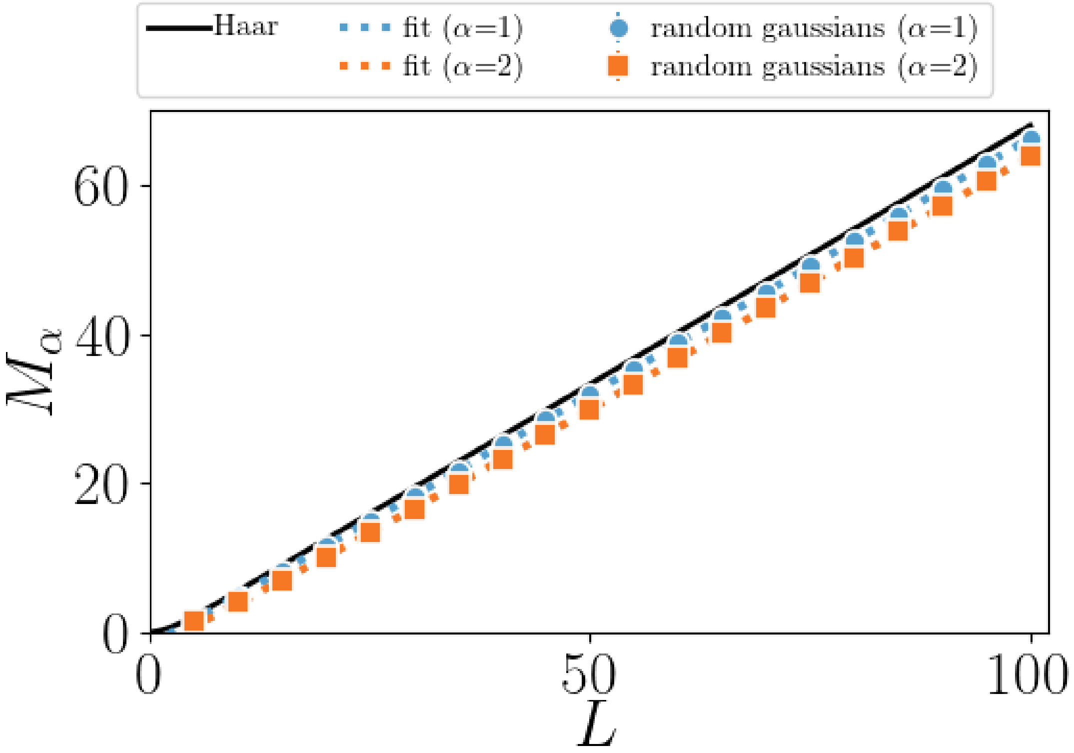
arxiv:2411......

1. How much quantum magic is stored in typical FGS?

Magic of Random FGS

$$\Gamma = O\Gamma(|0...0\rangle)O^T \qquad O \sim \text{Haar}[O(2L)]$$

(random) Gaussian transformation



Magic of Random FGS

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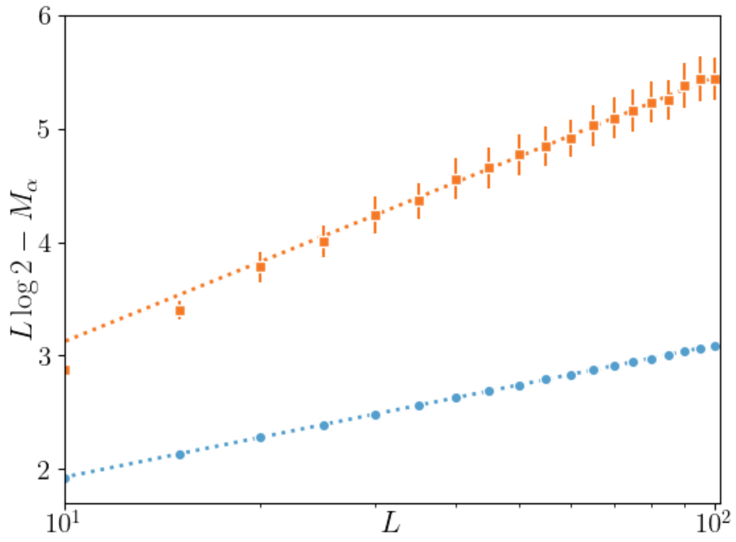
(random) Gaussian transformation



Our results

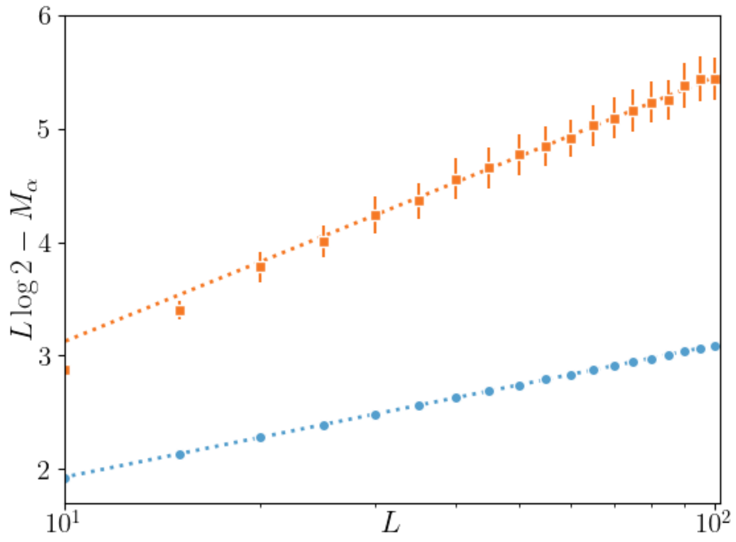
Numerically there is evidence that for Random FGS:

$$\mathcal{M}_\alpha = \overbrace{c(\alpha)L}^{\text{extensive part as Haar random states}} - \underbrace{a(\alpha)\log L}_{\text{logarithmic corrections}} + \text{cost.}$$

Magic of Random FGS

$$\Gamma = O\Gamma(|0...0\rangle)O^T \qquad O \sim \text{Haar}[O(2L)]$$

(random) Gaussian transformation



Our results

Analytics suggests that for random FGS:

$$\mathcal{M}_\alpha = \overbrace{c(\alpha)L}^{\substack{\text{extensive part} \\ \text{as Haar random states}}} - \underbrace{a(\alpha)\log L}_{\text{logarithmic corrections}} + \text{cost.}$$

Participation Entropies of Random FGS

Participation Rényi Entropies

$$I_n(|\psi\rangle) = \sum_{\boldsymbol{z}} |\langle \boldsymbol{z}|\psi\rangle|^{2n} \qquad \mathcal{S}_n = (1-n)^{-1} \log I_n(|\psi\rangle)$$

Our results

Analytical calculations for random FGS give:

$$\mathcal{S}_\alpha = \overbrace{c(\alpha) L}^{\substack{\text{extensive part} \\ \text{as Haar random states}}} - \underbrace{a(\alpha) \log L}_{\text{logarithmic corrections}} + \text{cost.}$$

Conclusions / Outlook

Matrix Product States

1. Typical MPS with bond dimension $\chi \sim$pol(N) are fully magic
2. MPS can be sampled efficiently to get accurate estimation of the SREs (up to $\sim O(100)$ qubits)

Free fermions

1. Typical FGS are fully magic, a part for corrections which are logarithmic in the system size
2. FGS can be sampled efficiently to get accurate estimation of the SREs (up to $\sim O(1000)$ fermionic modes)

Thank you!